

## "If" or "only if"

### Question 1

Given that 
$$\begin{cases} A_1x + B_1y + C_1 = 0 \\ A_2x + B_2y + C_2 = 0 \end{cases} \quad (1)$$

are two non-parallel straight lines and  $(x_0, y_0)$  be their intersection point.

Prove that every line passing through  $(x_0, y_0)$  is of the form :

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0 \quad (2)$$

### "Proposed" proof

$$(x_0, y_0) \text{ passes through } A_1x + B_1y + C_1 = 0 \quad \Rightarrow \quad A_1x_0 + B_1y_0 + C_1 = 0 \quad (3)$$

$$(x_0, y_0) \text{ passes through } A_2x + B_2y + C_2 = 0 \quad \Rightarrow \quad A_2x_0 + B_2y_0 + C_2 = 0 \quad (4)$$

Since  $A_1x_0 + B_1y_0 + C_1 + k(A_2x_0 + B_2y_0 + C_2) = 0 + k(0) = 0$  , by (3) and (4) .

therefore  $(x_0, y_0)$  lies on the straight line (2) .

In other words, the straight line (2) passes through  $(x_0, y_0)$  .

### Analysis

We have just proved that every line of the form (2) passes through  $(x_0, y_0)$  . We have not shown that every line passing through  $(x_0, y_0)$  is of the form (2).

### Correct proof

Every line passing through  $(x_0, y_0)$  must be of the form:  $y - y_0 = m(x - x_0)$  (5)

In order to show that (5) is of the form (2) , we have to find  $k$  . We choose  $m$  such that :

$$-m = \frac{A_1 + kA_2}{B_1 + kB_2} \quad \text{or} \quad k = \frac{A_1 - mB_1}{A_2 + mB_2} \quad (6)$$

This is always possible since the lines (1) are non-parallel and the denominators of (6) are non-zero .

Now, 
$$y - y_0 = m(x - x_0)$$

$$\Leftrightarrow y - y_0 + \frac{A_1 + kA_2}{B_1 + kB_2}(x - x_0) = 0$$

$$\Leftrightarrow (A_1 + kA_2)(x - x_0) + (B_1 + kB_2)(y - y_0) = 0$$

$$\Leftrightarrow (A_1 + kA_2)x + (B_1 + kB_2)y - (A_1 + kA_2)x_0 - (B_1 + kB_2)y_0 = 0$$

$$\Leftrightarrow (A_1 + kA_2)x + (B_1 + kB_2)y - (A_1x_0 + B_1y_0) - k(A_2x_0 + B_2y_0) = 0$$

$$\Leftrightarrow (A_1 + kA_2)x + (B_1 + kB_2)y + (C_1 + kC_2) = 0 \quad , \text{ since } (x_0, y_0) \text{ lies on lines (1) .}$$

$$\Leftrightarrow A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0$$

The proof is thus completed.

The problem becomes more complicate for coaxial circles, although the concept is similar. This is illustrated in the following question.

**Question 2**

Given that 
$$\begin{cases} C_1 : x^2 + y^2 + D_1x + E_1y + F_1 = 0 \\ C_2 : x^2 + y^2 + D_2x + E_2y + F_2 = 0 \end{cases} \quad (1)$$

are two intersecting circles and  $(x_1, y_1), (x_2, y_2)$  be their intersection points.

Prove that every circle passing through  $(x_1, y_1), (x_2, y_2)$  is of the form :

$$C: x^2 + y^2 + D_1x + E_1y + F_1 + k(x^2 + y^2 + D_2x + E_2y + F_2) = 0 \quad (k \neq -1) \quad (2)$$

**"Proposed" proof**

Obviously (2) is a circle since it can be rearranged as a form of a circle, that is:

$$(1+k)x^2 + (1+k)y^2 + (D_1 + kD_2)x + (E_1 + kE_2)y + (F_1 + kF_2) = 0, \text{ or}$$

$$x^2 + y^2 + \left(\frac{D_1 + kD_2}{1+k}\right)x + \left(\frac{E_1 + kE_2}{1+k}\right)y + \left(\frac{F_1 + kF_2}{1+k}\right) = 0 \quad (k \neq -1).$$

Now,  $(x_1, y_1)$  passes through  $x^2 + y^2 + D_1x + E_1y + F_1 = 0 \Rightarrow x_1^2 + y_1^2 + D_1x_1 + E_1y_1 + F_1 = 0 \quad (3)$

Also,  $(x_1, y_1)$  passes through  $x^2 + y^2 + D_2x + E_2y + F_2 = 0 \Rightarrow x_1^2 + y_1^2 + D_2x_1 + E_2y_1 + F_2 = 0 \quad (4)$

By (3) and (4),  $x_1^2 + y_1^2 + D_1x_1 + E_1y_1 + F_1 + k(x_1^2 + y_1^2 + D_2x_1 + E_2y_1 + F_2) = 0 + k \times 0 = 0$ , therefore  $(x_1, y_1)$  lies on the circle (2).

Similarly,  $(x_2, y_2)$  lies on the circle (2).

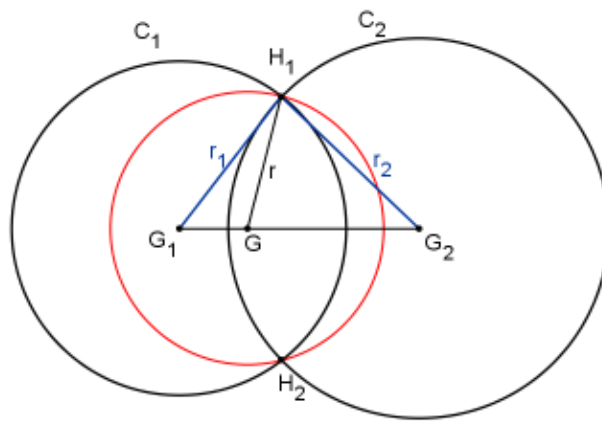
In other words, the circle (2) passes through  $(x_1, y_1), (x_2, y_2)$ .

**Analysis**

The same mistake arises. We have just proved that every circle of the form (2) passes through  $(x_1, y_1), (x_2, y_2)$ . We have not shown that every circle passing through  $(x_1, y_1), (x_2, y_2)$  is of the form (2).

**Correct proof**

The centre, G, of any circle C, with radius r, passing through the points  $H_1(x_1, y_1), H_2(x_2, y_2)$  must also lie on the line joining the centres  $G_1$  and  $G_2$  of the circles  $C_1$  and  $C_2$  respectively with equations given in (1).



$$G_1 = \left(-\frac{D_1}{2}, -\frac{E_1}{2}\right), \quad G_2 = \left(-\frac{D_2}{2}, -\frac{E_2}{2}\right) \quad (5)$$

Let  $r_1$  and  $r_2$  be the radii of the circles  $C_1$  and  $C_2$ .

$$r_1^2 = \left(\frac{D_1}{2}\right)^2 + \left(\frac{E_1}{2}\right)^2 - F_1, \quad r_2^2 = \left(\frac{D_2}{2}\right)^2 + \left(\frac{E_2}{2}\right)^2 - F_2 \quad (6)$$

Let  $G_1G : GG_2 = \lambda : (1-\lambda)$ , where  $-\infty < \lambda < +\infty$ .

By section formula,

$$\begin{aligned} G &= \left( \lambda \left( -\frac{D_1}{2} \right) + (1-\lambda) \left( -\frac{D_2}{2} \right), \lambda \left( -\frac{E_1}{2} \right) + (1-\lambda) \left( -\frac{E_2}{2} \right) \right) \\ &= \left( -\frac{\lambda D_1 + (1-\lambda)D_2}{2}, -\frac{\lambda E_1 + (1-\lambda)E_2}{2} \right) \end{aligned} \quad (7)$$

Let Radius of  $C = GH_1 = r$ , then from (5) and (7),

$$\begin{aligned} r^2 &= \left( x_1 + \frac{\lambda D_1 + (1-\lambda)D_2}{2} \right)^2 + \left( y_1 + \frac{\lambda E_1 + (1-\lambda)E_2}{2} \right)^2 \\ &= x_1^2 + y_1^2 + x_1(\lambda D_1 + (1-\lambda)D_2) + y_1(\lambda E_1 + (1-\lambda)E_2) + \left( \frac{\lambda D_1 + (1-\lambda)D_2}{2} \right)^2 + \left( \frac{\lambda E_1 + (1-\lambda)E_2}{2} \right)^2 \\ &= \lambda(x_1^2 + y_1^2 + D_1x + E_1y) + (1-\lambda)(x_1^2 + y_1^2 + D_2x + E_2y) + \left( \frac{\lambda D_1 + (1-\lambda)D_2}{2} \right)^2 + \left( \frac{\lambda E_1 + (1-\lambda)E_2}{2} \right)^2 \end{aligned} \quad (8)$$

Since  $H_1(x_1, y_1), H_2(x_2, y_2)$  are on the circles  $C_1$  and  $C_2$ ,

$$\begin{cases} x_1^2 + y_1^2 + D_1x_1 + E_1y_1 + F_1 = 0 \\ x_1^2 + y_1^2 + D_2x_1 + E_2y_1 + F_2 = 0 \end{cases}$$

$$\therefore (8) \text{ becomes, } r^2 = \left( \frac{\lambda D_1 + (1-\lambda)D_2}{2} \right)^2 + \left( \frac{\lambda E_1 + (1-\lambda)E_2}{2} \right)^2 - [\lambda F_1 + (1-\lambda)F_2] \quad (9)$$

Using centre-radius form, by (7) and (9) the equation of  $C$  is therefore,

$$\begin{aligned} \left( x + \frac{\lambda D_1 + (1-\lambda)D_2}{2} \right)^2 + \left( y + \frac{\lambda E_1 + (1-\lambda)E_2}{2} \right)^2 &= \left( \frac{\lambda D_1 + (1-\lambda)D_2}{2} \right)^2 + \left( \frac{\lambda E_1 + (1-\lambda)E_2}{2} \right)^2 - [\lambda F_1 + (1-\lambda)F_2] \\ x^2 + y^2 + x[\lambda D_1 + (1-\lambda)D_2] + y[\lambda E_1 + (1-\lambda)E_2] + [\lambda F_1 + (1-\lambda)F_2] &= 0 \\ \lambda(x^2 + y^2 + D_1x + E_1y + F_1) + (1-\lambda)(x^2 + y^2 + D_2x + E_2y + F_2) &= 0 \end{aligned} \quad (10)$$

For  $\lambda = 0$ , (10) is reduced to the circle  $C_2$ , assuming  $\lambda \neq 0$ , we have:

$$x^2 + y^2 + D_1x + E_1y + F + \left( \frac{1-\lambda}{\lambda} \right) (x^2 + y^2 + D_2x + E_2y + F_2) = 0$$

By putting  $k = \frac{1-\lambda}{\lambda}$ , we can prove that :

Every circle passing through  $(x_1, y_1), (x_2, y_2)$  is of the form :

$$x^2 + y^2 + D_1x + E_1y + F_1 + k(x^2 + y^2 + D_2x + E_2y + F_2) = 0 .$$

The proof is thus completed.

### Lesson

Some problem or theorem has a "if" part and/or "only if" part, make sure that you know the direction you need to carry out your steps.